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# Vibrational spectra of diagonally self-affine fractals

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**Abstract.** The vibrational densities of states for a deterministic self-affine aggregate and the two-dimensional Eden hulls are computed numerically by the continued-fraction recursion method. It is shown that in contrast with the geometrical complexity the density of states for the diagonally self-affine fractals exhibits a simple power-law scaling. Motivated by these suggestive results, we extend the work done by Alexander and Orbach in 1982 and conclude that the concept of fractons is still valid in characterizing the vibrational spectra of all diagonally self-affine fractals. We also relate the fracton dimensionality to other exponents.

## 1. Introduction

Many recent studies have shown that a wide class of processes leads to complex objects which can be described in terms of self-affine fractals [1–3]. Examples range from plots of various kinds of random walk [1, 2] to interfaces developing in marginally stable, far-from-equilibrium systems [3–5]. As far as we know, all the work on self-affine fractals concentrates on the geometrical description [6–10]. It is shown that, in contrast with the unique fractal dimension of self-similar fractals, one needs in general several distinct notions. Most important are the concepts of local dimension and global dimension, valid on scales well below and well above, respectively, a certain crossover scale [6, 7]. For a self-affine fractal with a crossover scale close to the smallest length scale in the system, no local fractal dimension exists [3, 6, 7]. On the other hand, all the work in fracton dynamics is concerned with self-similar fractals [11–19]. It is shown that the vibrational density of states (VDOS) for self-similar fractals scales with frequency as $n(\omega) \sim \omega^{d_s-1}$, where $d_s$ is called the fracton dimensionality [11, 12]. So, it is of great value to study the dynamical behaviour of self-affine fractals.

In this paper, we wish to report the first results for the vibrational behaviour of self-affine fractals, and in particular we wish to answer the questions of whether the concept of fractons is still valid in characterizing the density of states (DOS) of diagonally self-affine fractals and how the fracton dimensionality is related to other exponents. These questions arise easily when one begins to deal with the dynamics of self-affine systems since the concept of fractons is introduced from studies on

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self-similar fractals [11, 12]. The fracton DOS (i.e. the simple power-law scaling form $n(\omega) \sim \omega^{d_s - 1}$) in self-affine structures is by no means more obvious than it seems to be at first sight. There are two reasons. First, since it needs more than one parameter to characterize the geometry of self-affine fractals, it is obvious that the physics of self-affine fractals are more complicated than those of self-similar fractals. Although much work has been devoted to the vibrational properties of self-similar fractals [11–19], the problem for self-affine structures is still open and it cannot be considered to be analogous to that of self-similar fractals without any investigation. Second, as we know, the power-law scaling form of the fracton DOS is based upon two assumptions: the vibration is scalar and the structure is self-similar [11, 12]. Several workers [20–22] have shown that, if the vector nature of the vibration is taken into account, the DOS will deviate from the power-law scaling. Therefore it is worthwhile to study the results of the replacement of self-similarity by self-affinity.

In the following section we present the numerical results of the VDOSs for two typical self-affine fractals and in section 3 we give an analytical argument. A discussion is presented in section 4 and this paper is closed with a summary in section 5.

## 2. Numerical results

We consider the vibration on diagonally self-affine networks whose structures are invariant under dilation transformation only if the lengths are rescaled by direction-dependent factors. Our results will apply to frequencies such that the associated distances are much larger than the size of the individual bonds making up the network, but much smaller than the total size of the network. The first self-affine fractal studied here is a two-dimensional deterministic aggregate [23] shown in figure 1. It can be grown by an iterative method. First, a seed particle is centred at the origin, and then one adds six particles to the seed to construct the cluster in the second iteration. At the $n$th iteration, the $(n-1)$th aggregate is taken to be a unit, and six such units are added to the $(n-1)$th aggregate: two in directions $\pm x$, and one in directions $\pm y$. The self-affine structure is produced in the $n \to \infty$ limit. Since the width of the structure grows with $n$ as $3^n$ while its length increases as $5^n$, the global dimension of this aggregate equals unity (here we use box dimensions). This self-affine fractal has a lower cut-off length which is the size of the particle that it is made of; therefore, it has no local dimension [3].

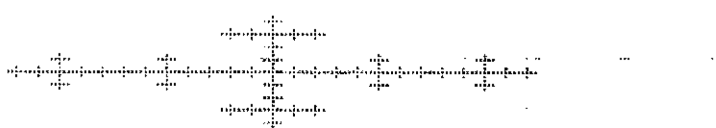

**Figure 1.** A two-dimensional deterministic self-affine aggregate.

The elastic lattice Hamiltonian takes the form of the scalar [24] Born model [25]:

$$H = \frac{\beta}{2} \sum_{\substack{ij \\ \mathrm{NN}}} (u_i - u_j)^2 \qquad (1)$$

where $u$ is the scalar displacement and the summation is over all nearest-neighbour (NN) particles. To compute the VDOS, we adopt the widely used recursion method of Haydock *et al* [26] (see also [27]) which offers fast computational speed and, above all, reliability. We choose an initial vector with all elements as random variables chosen from a Gaussian distribution [21]. Here ten initial vectors have been averaged to obtain the global VDOS and the length of the continued fraction is set to $L_L = 100$. Figure 2 shows the *integrated* VDOS $N(\omega) = \int_0^\omega n(\omega')\, d\omega'$ for the self-affine aggregate of figure 1 with cluster size $N = 16807 (= 7^5)$. The integrated VDOS at lower frequencies follows a power law $N(\omega) \sim \omega^{d_s}$ down to a finite-size-induced frequency $\omega_{min}$. Since the system is not fractal on length scales shorter than the lattice spacing (we take it to be unity), $N(\omega)$ does not keep its form above $\omega \simeq 2\pi$. The value of $d_s$ is obtained by fitting a least-squares line over the range from $\log_{10} \omega = -1.1$ to $\log_{10} \omega = 0.30$, and $d_s = 1.19 \pm 0.02$.

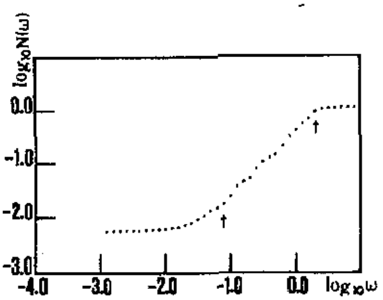


**Figure 2.** Log–log dependence of the integrated VDOS on the frequency for the self-affine aggregate shown in figure 1. The fracton dimensionality is obtained by fitting a least-squares line between the two points indicated by arrows.

The second self-affine fractals are the rough surfaces of the Eden clusters. We first grew the two-dimensional Eden clusters in the strip geometry using version C of Jullien and Botet [28]. After a sufficiently long growth 'time', the surfaces will become stationary, i.e. independent of the height ('time') of the surfaces [29]. At this time the rough surfaces are statistically self-affine [3]. We then determined the external 'hull' [30] of the Eden clusters. A typical hull is shown in figure 3. The last step is to compute the VDOSs of the Eden hulls using the fraction recursion method [26, 27] mentioned above with a periodic boundary condition along the substrate. Figure 4 shows the log–log plot of the integrated VDOSs versus frequency for the Eden hulls at the stationary stage (numerically, for height $H \geqslant 40L$, where $L$ is the length of the substrate). Each result is an average over 100 configurations. The data set are shifted vertically for clarity of display. It is shown from the log–log linearity that the integrated VDOSs scale with frequency as a power law $N(\omega) \sim \omega^{d_s}$. Fitting a least-square line over the range from $\log_{10} \omega = -0.4$ to $\log_{10} \omega = 0.1$ we obtain that $d_s$ is around $1.55 \pm 0.08$, independent of the growth time when the surface is geometrically stationary (we found that the growth dynamics come into the vibrational properties only at the early growth 'time'). The uncertainty in $d_s$ is fairly large since the strip length $L$ is not large and the computer time prevents us from growing clusters using a larger $L$.

**Figure 3.** A typical hull of the two-dimensional Eden model in a strip geometry.

The above numerical results indicate that the vibrational spectra for the two self-affine fractals are characterized by the fracton DOS $n(\omega) \sim \omega^{d_s - 1}$, where the fracton dimensionalities $d_s$ take non-trivial values.

### 3. Analytical argument

Motivated by the numerical results, we further wish to know whether the fracton DOS for self-affine fractals is generally valid and how the fracton dimensionality is related to other exponents. The following argument is a straightforward extension of the work of Alexander and Orbach [11] who first proposed the concept of fractons. The power-law behaviour of the VDOS can be obtained using a Green function technique. The DOS for the vibration problem can be mapped onto the DOS for the diffusion problem which can be obtained from the single-site Green function:

$$n(\omega) = -(1/\pi)\,\mathrm{Im}\langle \tilde{P}_0(-\epsilon + \mathrm{i}0^+)\rangle \tag{2}$$

where $n(\omega)$ is the VDOS and $\tilde{P}_0(\epsilon)$ is the Laplace transformation of $P_0(t)$, the autocorrelation function, with $\epsilon$ the spectral parameter. On self-affine fractals, one expects, in general, anomalous diffusion along different directions:

$$\langle r_\mu^2(t)\rangle \sim t^{2/(2+\theta_\mu)} \qquad \mu = 1, 2, \ldots, d. \tag{3}$$

The total volume on the diagonally self-affine fractal within the diffusion distance is generally [23]

$$V(t) \sim \sum_{\mu=1}^{d} A_\mu \left[r_\mu^2(t)\right]^{D_\mu/2} \tag{4}$$

where $\{D_\mu\}$ are the characteristic exponents governing the mass–length scaling. Thus,

$$\langle P_0(t)\rangle \sim [V(t)]^{-1} \sim \left(\sum_{\mu=1}^{d} A_\mu t^{D_\mu/(2+\theta_\mu)}\right)^{-1}. \tag{5}$$

As $t \to \infty$, $P_0(t) \sim t^{-d_s/2}$, with

$$d_s = \max_{1\leqslant\mu\leqslant d}(2D_\mu/(2+\theta_\mu)). \tag{6}$$

Then the VDOS exhibits a power-law scaling

$$n(\omega) \sim \omega^{d_s-1} \tag{7}$$

where $d_s$ is called the fracton dimensionality.

### 4. Discussion

For self-affine fractals the quantities $2D_\mu/(2+\theta_\mu)$ in equation (6) are, in general, different for different $\mu$. However, here we cite an example for which these quantities are the same for different $\mu$. It is one of the single-valued self-affine curves $M_{\mathrm{H}}$ with the Holder exponent $H = \frac{1}{2}$ [3, 6]. This curve is a deterministic version of a Brownian plot and was first invented by Mandelbrot [6]. It can be constructed iteratively. Figure 5 shows its first three generations (for construction, see [3]). In the $k$th stage

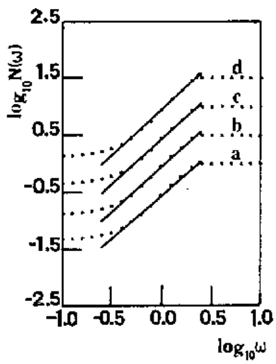


**Figure 4.** Log–log plot of the integrated VDOSS versus frequency for the Eden hulls at the stationary growth stage for a strip width $L = 196$ and various growth heights: data a, $H = 40L$; data b, $H = 50L$; data c, $H = 60L$: data d, $H = 70L$.

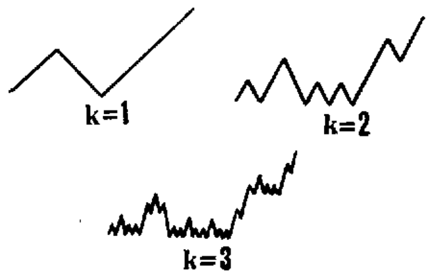


**Figure 5.** The first three generations of a self-affine curve.

we obtain $M_{\mathrm{H}}^{(k)}$. If $k$ is finite, the curve will be composed of many bonds. Such bonds are assumed to connect pairs of particles which vibrate with the Hamiltonian in equation (1). It is easy to calculate the exponents $D_\mu$ and $\theta_\mu (\mu = 1, 2)$ for this curve. If one scales the lengths in the $x$ direction by a factor of 4 and in the $y$ direction by a factor of 2, we shall find that the number of bonds (or particles) increases four times. Therefore, $D_1 = (\ln 4)/(\ln 4) = 1$ and $D_2 = (\ln 4)/(\ln 2) = 2$. Topologically, the curve is equivalent to a line. When a particle travels a distance $l \sim \sqrt{t}$ along the self-affine line, it will travel a Euclidean distance with components $x \sim l \sim \sqrt{t}$, $y \sim \sqrt{l} \sim t^{1/4}$. From (3) we have $\theta_1 = 0$, $\theta_2 = 2$. Therefore, $2D_1/(2 + \theta_1) = 2D_2/(2 + \theta_2) = 1$, and from (6) we have $d_s = 1$.

It can be demonstrated that $d_s = 1$ is correct if it is noted that the Hamiltonian in equation (1) (and therefore the vibrational properties) for this self-affine curve is the same as that of a line!

Our second example in section 2 is the Eden hull whose $d_s$ is found to be about 1.55. At first sight, one may wonder why its fracton dimensionality $d_s$ is not equal to unity as in the case in figure 5. This is because the Eden hulls are not topologically equivalent to a line. The environment of each particle in the Eden hulls is different from that in a line and this difference cannot be smeared out in the asymptotic regime. Topologically one can extract a line from the Eden hull and the particles can be classified into two types: particles on the 'line' and particles not on the 'line'. Suppose that there are $N_1$ and $N_2$ of these two types of particle, respectively, in the asymptotic regime $N = N_1 + N_2 \to \infty$; it is most probable that $N_1/N$ and $N_2/N$ approach two non-zero constants. For the Eden hulls grown from a finite substrate with length $L = 196$, we have estimated that $N_2/N$ is of the order of 10% and we conjecture that this value is also valid in the $N \to \infty$ limit. It is this non-zero $N_2/N$ fraction of particles that leads to a $d_s$-value somewhat larger than unity.

Generally, the exponent $d_s$ in (6) may take a non-trivial value, i.e. different from the value of the Euclidean dimensions. From (6) and (7) one can conclude that the concept of fracton dimensionality is generally valid in characterizing the VDOS for diagonally self-affine fractals.

## 5. Summary

We have shown numerically that the VDOSs for a deterministic self-affine aggregate and the two-dimensional Eden hulls are characterized by a simple power-law behaviour although their geometry is different from self-similarity and the fracton dimensionalities are found to have non-trivial values. The numerical results indicate that fractons exist in the self-affine fractals that we studied and shed light on the dynamical behaviours of other self-affine fractals. A straightforward extension of the work of Alexander and Orbach confirms our numerical findings and relates the fracton dimensionality to other exponents. It is interesting to investigate the vibrational properties of non-diagonally self-affine fractals.

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